THE STONE-WEIERSTASS THEOREM

ANDREA FERRETTI

The aim of this note is to write down a proof of the celebrated Stone-Weierstrass theorem. Many proofs are now available, but this particular one strikes me for its simplicity and its clever use of almost all basic theorems in functional analysis; indeed it is one of my favourite proofs in all mathematics. Sadly I have lost trace of where I read it originally, so I repeat it here to preserve it.

Theorem 1 (Stone-Weierstrass)**.** *Let X be a compact Hausdorff space and let $\mathcal{A} \subset C(X) = C(X, \mathbb{R})$ be an algebra of continous functions such that:*

i) for every distinct $x, y \in X$ there exists a function $f \in A$ such that $f(x) \neq f(y)$ (we say that $\mathcal{A}$ separates points);

ii) for every $x \in X$ there exists a function $f \in \mathcal{A}$ such that $f(x) \neq 0$.

Then $\mathcal{A}$ is dense in $C(X)$.

Note that *ii*) holds whenever $\mathcal{A}$ contains the constant functions, and often the theorem is formulated with this assumption in place of *ii*).

Proof. First note that the closure of $\mathcal{A}$ is still an algebra, so we can assume that $\mathcal{A}$ is closed and prove the stronger thesis that $\mathcal{A} = C(X)$. Assume the contrary; then by the Hahn-Banach theorem we can find a functional $\phi \in C(X)^*$ such that $\phi = 0$ on $\mathcal{A}$. So it is enough to prove that the annihilator $\mathcal{A}^\perp \subset C(X)^*$ is trivial.

Let $B \subset \mathcal{A}^\perp$ be the closed unit ball; of course it is enough to prove that $B = \{0\}$. Endow $C(X)^*$ with the weak-$*$ topology; by the Banach-Alaoglu theorem B is compact. Since it is also convex, it is the closed convex hull of its extremal points, by Krein-Milman theorem. So it is enough to prove that any extreme point of B is the 0 functional.

To prove the last claim we use the Riesz-Markov theorem to identify $C(X)^*$ with the space of (signed) Radon measures on X. So let μ be an extreme point of B. In particular if $\mu \neq 0$ we have $|\mu|(X) = 1$. Note that for any $f \in A$ the measure μ_f defined by

$$\mu_f(U) = \int_U f d\mu$$

is again in $\mathcal{A}^\perp$. Indeed for any $g \in \mathcal{A}$

$$\int_X g d\mu_f = \int_X fg d\mu = 0$$

since $fg \in \mathcal{A}$. Let us use the notation

$$\mu(f) := \int_X f d|\mu|.$$

Then for any $f \in \mathcal{A}$ with $\mu(f) \neq 0$ we have $\mu_f/\mu(f) \in B$.

Note that $\mu(f) + \mu(1-f) = 1$ and

$$\mu = \mu(f)\frac{\mu_f}{\mu(f)} + \mu(1-f)\frac{\mu_{1-f}}{\mu(1-f)}.$$

So if $\mu(f) \in]0, 1[$, since μ is extreme in B, we deduce that either $\mu = \mu_f$ or $\mu = \mu_{1-f}$.

Date: July 3, 2010.

From this we deduce that the support of μ consists of a single point. Indeed assume that we can find two distinct points $x, y \in \mathit{supp}\,\mu$. By assumption *i*) we can find a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Moreover we can assume that $\mu(f) \in]0,1[$ (a suitable linear combination of f and f^2 will do), hence either $\mu = \mu_f$ or $\mu = \mu_{1-f}$. By symmetry assume the first; since f is continous μ differs from μ_f on a suitably small open set either around x or around y, contradiction.

Let $\mathit{supp}\,\mu = \{x\}$; then μ is a multiple of δ_x. But then for every $f \in \mathcal{A}$ we have

$$f(x) = \alpha \int_X f d\mu = 0$$

since $\mu \in \mathcal{A}^{\perp}$. This contradicts assumption *ii*). $\square$

E-mail address: `ferrettiandrea@gmail.com`